A THEORETICAL STUDY OF ZONALLY UNIFORM OCEANIC FLOW

by

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ABSTRACT

A mathematical model of a zonally uniform ocean is constructed in order to investigate the steady circulation and the nonhomogeneous distribution of density maintained by elementary thermodynamic processes. The thermodynamic processes are considered implicitly by introducing an equation to represent the transfer of mass by convection and diffusion. The thermodynamic processes which affect the density in the surface layers of the ocean are replaced by a distribution of equivalent mass flux across the surface.

A simplified system of equations is derived from the general system of hydrodynamic equations representing the model by neglecting small terms in the equations. The simplified system of equations is nonlinear and of the boundary-layer type. An approximate analysis is carried out using simplified boundary-layer methods and the following types of flow with their associated density fields are studied in detail: (A) the flow set up in a homogeneous ocean by surface wind stress, (B) the flow and the density field set up by a given mass flux across the ocean surface in the absence of wind stress, (C) the flow and the density field set up by a given mass flux across the ocean surface in the presence of wind stress. For purposes of comparison, the same distributions of wind stress and surface mass flux are used in studying the three different types of flow.

The different types of flow are illustrated by diagrams showing the streamlines of the meridional circulation and the distribution of isopycnals in a meridional section. The vertical distributions of the zonal velocity component for the different types of flow are also compared.

From the analysis of this model, it is concluded that thermodynamic processes affecting the density of sea water cannot be neglected in the study of zonally uniform currents in the ocean. Thus, it does not appear to be possible to account for the observed features of an ocean current, such as the Antarctic Circumpolar Current, in terms of surface wind stress alone.

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1. Introduction

The dynamics of atmospheric and oceanic circulation has been studied extensively in the past with the result that many theoretical models based on the hydrodynamic equations have been constructed to interpret various aspects of the balance of forces achieved in large-scale fluid motion. The study of the thermodynamic processes by which some of these forces are created and maintained has not been very successful from a theoretical point of view because of the difficulties involved in treating these processes analytically. As a consequence, discussions of the effects of thermodynamic processes on large-scale fluid motion have been qualitative in nature. However, an adequate understanding of atmospheric and oceanic circulation cannot be achieved until the baroclinic structure of the atmosphere and ocean can be interpreted quantitatively in terms of suitable models which include thermodynamic processes.

The purpose of the present investigation is to consider a theoretical model of a zonally uniform ocean, i.e., an ocean having properties that do not vary with longitude. Thermodynamic processes are introduced in a sufficiently simple manner to allow analytical treatment of the model.

Many investigations of atmospheric and oceanic circulation have been made using models based on zonally uniform flow, so that a considerable amount of literature on the dynamics of zonally uniform flow is available. The classical Hadley model of atmospheric circulation is an example. It is related to the present model in that the qualitative effects of thermodynamic processes are considered. Rossby (1941) has described the basic concepts as well as the limitations of the Hadley model. Attempts have been made to construct a quantitative Hadley model. Dorodnitsyn, Izvekov, and Schwetz (1939), for example, constructed a semi-empirical model of the Hadley type using observed distributions of temperature and surface pressure in the atmosphere; however, they avoided explicit discussion of the thermodynamic processes which accompany the resulting atmospheric circulation.

The thermodynamic processes occurring in the ocean are recognized to be of importance in their effect on the circulation, particularly in the polar regions. Deacon (1933) and Sverdrup (1933) indicate that thermodynamic processes play an important role in determining the behavior of the Antarctic Circumpolar Current (ACC).

The ACC is the prototype of the present model in that the dimensions and the range of variation of the various quantities considered in the model are chosen to be of the order of magnitude observed in the ACC. However, the ACC is not zonally uniform. It has been suggested by Munk and Palmén (1951) that the departure from zonal uniformity may be necessary for a dynamic equilibrium to be achieved. As all zonal variations are excluded in the present model, the results obtained probably have limited applicability to actual conditions existing in the ACC.

2. General problem

The problem considered here is, in its most general sense, that of determining the circulation and the distribution of physical and chemical properties such as density, temperature, salinity, etc., produced by a given surface wind stress and a given flux of heat and water across the ocean surface. Only a small portion of the general problem is actually considered and many simplifying assumptions must be made before explicit results can be obtained.

At least two fundamental difficulties are encountered in setting up a theoretical model to include thermodynamic processes, and these difficulties are avoided rather than resolved in the present investigation. The first difficulty is encountered in setting up the dynamic portion of the model. Theoretically, one might expect that the torque of the zonal component of surface wind stress is either transmitted vertically to the ocean bottom or laterally to the continental coasts. If the torque is transmitted to the ocean bottom, there are strong currents near the bottom. There is no conclusive observational evidence to indicate whether or not such bottom currents exist. Munk and Palmén (1951) say that lateral stresses in the ACC do not appear to be large enough to balance the wind stress. They suggest that the balance could be accomplished by the "mountain effect" of submarine ridges which, in effect, produce a retarding torque on the ACC. The "mountain effect" cannot be incorporated into the present model except by postulating ad hoc a zonal force of an arbitrary nature. The present model is set up so that the zonal component of wind stress is transmitted almost entirely to the ocean bottom.

The second difficulty is encountered in introducing thermodynamic processes. The important thermodynamic processes which affect the density of the water act in the surface layers of the ocean. It has not been found possible to consider processes which tend to increase the density near the surface of the ocean. If such processes are considered, the resulting density distribution is unstable, i.e., the density increases upward, in the region where these processes occur. This difficulty is avoided by considering only those thermodynamic processes whose net effect is to lower the density near the ocean surface and by limiting the discussion to an interior region of the ocean away from continental coasts. It is possible then to consider a steady state in a limited region of the ocean by regarding the ocean outside of this region as a reservoir which can supply water, heat, salt, etc., at a rate necessary to maintain the steady state.

3. Simplifying assumptions

The basic assumptions necessary to set up the mathematical equations of the model are given below. More specific assumptions are discussed as they arise in the analysis of the model.

The ocean basin is taken to be zonally uniform and of constant depth with a level bottom. Lateral boundaries are not explicitly considered. The coordinates of a point in the ocean can be specified by the latitude, longitude, and height above the ocean bottom measured along the local vertical. It is assumed that this coordinate system can be replaced with sufficient accuracy by a spherical coordinate system (λ, ϕ, r) in which gravity acts in the radial direction. Here, λ is the longitude, ϕ the latitude, and r the distance of the point from the center of the earth. The distance r is approximated by a + z, where a is the mean radius of the earth and z is the height above the ocean bottom.

The velocity components relative to the earth (u, v, w) are chosen so that u is the eastward component, v the northward component, and w the upward component.

Because the density variations in the ocean are a small fraction of the total density, the variations are neglected in the equations of motion except in the term representing the force of gravity. The compressibility of the water is neglected everywhere, so that a stream function can be introduced for the circulation in the meridional plane. The flux of water across the ocean surface is negligible as far as the stream function is concerned, because the amount of water exchanged with the atmosphere by evaporation and precipitation is small in comparison with the amount of water transported by the meridional currents. However, the effect of the flux of water on the density cannot be neglected because the water added is generally of a different density. In its effect on the density, the flux of water can be interpreted as a flux of mass numerically equal to the difference in mass between the volume of water added across a unit area of ocean surface per unit time and an equal volume of oceanic water. Similarly, the volume change due to thermal expansion is not significant in comparison with the total volume of water in circulation but the density change due to the thermal expansion is dynamically significant. The heat flux must be included in terms of an equivalent mass flux across the surface of the ocean.

The diffusion and convection of heat, salt, etc., in the interior of the ocean are not taken into account separately, but their effect is taken into account by introducing an equation to represent the diffusion and convection of mass. The assumptions under which this simplification is justified are given in Appendix I, which includes the derivation of the mass-transfer equation.

Instead of using nonisotropic eddy coefficients having large values to characterize horizontal transfers of momentum and mass, the nonlinear terms representing convective transfers of these quantities are retained in the equations and the diffusive processes are assumed to be characterized by constant, isotropic kinematic eddy coefficients. The kinematic eddy coefficients of viscosity and diffusivity are assumed to be of different orders of magnitude for reasons given in Appendix II.

A basic assumption underlying the entire development of the model is that the circulation produced by steady, zonally uniform distributions of wind stress and mass flux is steady and zonally uniform. The dynamic stability of the circulation is, therefore, not investigated and the flow is treated as being stable.

4. Equations of motion

On the basis of the preceding assumptions, the equations governing the steady, zonally uniform motion and the distribution of density in the ocean may be written

$$\frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial r} + \frac{uw}{r} - \frac{\tan\phi\, uv}{r} + 2\Omega(\cos\phi\, w - \sin\phi\, v) = K\left(\nabla^2 u - \frac{u}{r^2\cos^2\phi}\right) \qquad (1)$$

$$\frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial r} + \frac{vw}{r} + \frac{\tan\phi\, u^2}{r} + 2\Omega\sin\phi\, u = -\frac{1}{\bar{\rho} r}\frac{\partial p}{\partial \phi} + K\left(\nabla^2 v - \frac{v}{r^2\cos^2\phi} + \frac{2}{r^2}\frac{\partial w}{\partial \phi}\right) \qquad (2)$$

$$\frac{v}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial r} - \frac{u^2+v^2}{r} - 2\Omega\cos\phi\, u = -\frac{1}{\bar{\rho}}\frac{\partial p}{\partial r} - \frac{\rho g}{\bar{\rho}} + K\left(\nabla^2 w - \frac{2w}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi} + \frac{2\tan\phi\, v}{r^2}\right) \qquad (3)$$

$$\frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\, v) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 w) = 0 \tag{4}$$

$$\frac{v}{r}\frac{\partial\rho}{\partial\phi} + w\frac{\partial\rho}{\partial r} = K_\rho \nabla^2 \rho, \tag{5}$$

where Ω is the earth's angular velocity, ρ the density considered as a function of the coordinates, $\bar{\rho}$ a constant mean density, and p the pressure. The kinematic eddy coefficients of viscosity and diffusivity are designated by K and K_ρ respectively, and gravity by g. The Laplacian in this coordinate system is

$$\nabla^2 = \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right).$$

A stream function ψ is introduced for the meridional circulation so that

$$v = \frac{1}{r\cos\phi}\frac{\partial\psi}{\partial r}$$

$$w = -\frac{1}{r^2\cos\phi}\frac{\partial\psi}{\partial\phi}.$$

The quantities M, x, D^2, and G are introduced and defined as follows:

$$M = r\cos\phi\, u,$$

the unit-mass angular momentum of the zonal flow,

$$x = \sin\phi$$

$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{1-x^2}{r^2}\frac{\partial^2}{\partial x^2}$$

$$G = x\frac{\partial}{\partial r} + \frac{1-x^2}{r}\frac{\partial}{\partial x}.$$

The differential operators D^2 and G are commutative. The operator G represents differentiation parallel to the earth's axis of rotation.

Equations (1) to (5) can be expressed more compactly in terms of these quantities as follows:

$$\frac{1}{r^2}\frac{\partial(\psi,M)}{\partial(r,x)} - 2\Omega G(\psi) = KD^2(M) \tag{6}$$

$$\frac{(1-x^2)^{\frac{1}{2}}}{r^2}\frac{\partial}{\partial(r,x)}\left[\psi,(1-x^2)^{-\frac{1}{2}}\frac{\partial\psi}{\partial r}\right] + \frac{xM^2}{r^2(1-x^2)} + 2\Omega xM$$
$$= -(1-x^2)\frac{\partial}{\partial x}\frac{p}{\bar{\rho}} + K\frac{\partial}{\partial r}D^2(\psi) \tag{7}$$

$$-\frac{1}{r^2}\frac{\partial}{\partial(r,x)}\left(\psi,\frac{1}{r^2}\frac{\partial\psi}{\partial x}\right) - \frac{1}{r^3(1-x^2)}\left[\left(\frac{\partial\psi}{\partial r}\right)^2 + M^2\right] - 2\Omega\frac{M}{r}$$
$$= -\frac{\partial}{\partial r}\frac{p}{\bar{\rho}} - \frac{\rho g}{\bar{\rho}} - \frac{K}{r^2}\frac{\partial}{\partial x}D^2(\psi) \tag{8}$$

$$\frac{1}{r^2}\frac{\partial(\psi,\rho)}{\partial(r,x)} = K_\rho\nabla^2\rho, \tag{9}$$

where

$$\frac{\partial(\psi,f)}{\partial(r,x)} \equiv \frac{\partial\psi}{\partial r}\frac{\partial f}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial f}{\partial r}$$

is the Jacobian of the functions ψ and f with respect to r and x.

Elimination of the pressure from (7) and (8) yields

$$\frac{1}{r^2}\frac{\partial}{\partial(r,x)}[\psi,D^2(\psi)] + \frac{2}{r^2(1-x^2)}[D^2(\psi)G(\psi) + MG(M)] + 2\Omega G(M)$$
$$= (1-x^2)\frac{\partial}{\partial x}\frac{\rho g}{\bar{\rho}} + KD^4(\psi). \tag{10}$$

The boundary conditions to be satisfied along the bottom of the ocean, $r = a$, are that all components of the velocity are zero and that the mass flux through the bottom is zero. Along the surface of the ocean, $r = a + h$, the boundary conditions are that the surface stress must equal the given wind stress and that the mass flux across the surface must equal the given mass flux.

Further restrictions must be made in order for a steady state to exist. The steady-state restrictions are that there can be no net volume transport across any latitude circle and that the net flux of angular momentum and of mass into the ocean must be zero.

The boundary conditions may be expressed mathematically as follows:

along the ocean bottom, $r = a$,

$$\partial\psi/\partial r = \partial\psi/\partial x = 0$$

$$M = 0$$

$$\partial\rho/\partial r = 0,$$

and along the ocean surface, $r = a + h$,

$$\partial\psi/\partial x = 0$$

$$Kr^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi}{\partial r}\right) = \frac{r(1 - x^2)^{\frac{1}{2}}}{\bar{\rho}} \tau_m$$

$$Kr^2 \frac{\partial}{\partial r}\frac{M}{r^2} = \frac{r(1 - x^2)^{\frac{1}{2}}}{\bar{\rho}} \tau = \frac{1}{\bar{\rho}} T$$

$$K_\rho \frac{\partial\rho}{\partial r} = \frac{1}{\bar{\rho}} Q.$$

Here, τ_m and τ are the meridional and zonal component respectively of the surface wind stress, T is the torque, about the earth's axis of rotation, produced by the surface wind stress, and Q is the mass flux across the surface of the ocean.

The steady-state restrictions on ψ, T^*, and Q^* are

$$\iint \psi \underset{\sim}{n} da = \iint T^* \underset{\sim}{n} da = \iint Q^* \underset{\sim}{n} da = 0,$$

where T^* and Q^* are the zonal torque and mass flux respectively. The surface integrals are taken over the entire ocean boundary, with $\underset{\sim}{n}$ denoting a unit vector, normal to the boundary, directed into the ocean. As the boundary of the ocean must be a streamline, the integral condition on ψ may be replaced by the condition $\psi = 0$ along the boundary. If the mass flux through the ocean bottom is zero, the condition on Q^*, for an ocean lying between x_1 and x_2, reduces to

$$\int_{x_1}^{x_2} Q dx = 0.$$

Thus, if the surface mass flux is different from zero, it must have both positive and negative values to satisfy the steady-state condition.

5. Dimensional analysis

The mathematical problem set up in the previous section is too complex to solve, so that further simplification is necessary. This simplification is achieved by neglecting terms which are estimated to be small in the equations representing the model. An estimate of the relative size of the various terms is made by assigning characteristic magnitudes to the dimensional quantities entering into the equations.

Accurate estimates can be made of the earth's mean radius a, the earth's rate of rotation Ω, the mean depth of the ocean h, gravity g, and the mean density $\bar{\rho}$. The magnitudes of the surface wind stress τ, the surface mass flux Q, and the fractional range of density variation γ can be estimated roughly from observations taken in the ocean. No satisfactory estimates are available from observations as to the magnitudes of the kinematic eddy coefficients K and K_ρ.

The equations representing the present model can best be studied by transforming them into nondimensional form. By studying the nondimensional form of the equations, it is possible to evaluate the effects of both large and small values of the eddy coefficients.

The dependent and independent variables are converted into nondimensional form, denoted by primes, through the following transformation:

$$r = aq' = a(1 + \alpha z'), \quad \alpha = h/a$$

$$x = x'$$

$$\rho = \overline{\rho}(1 + \gamma\rho')$$

$$\psi = \Psi\psi'$$

$$M = aUM'$$

$$T = a\tau_M T'$$

$$\tau_m = \tau_M \tau_m'$$

$$Q = Q_M Q',$$

where Ψ and U are characteristic values of the meridional stream function and the zonal velocity component respectively, τ_M and Q_M are maximum values of the wind stress and surface mass flux respectively.

The equations obtained after performing the transformation have the form

$$\frac{\Psi U}{ah}\frac{1}{q'^2}\frac{\partial(\psi',M')}{\partial(z',x')} - \frac{2\Omega\Psi}{h}G'(\psi') = \frac{KaU}{h^2}D'^2(M') \tag{11}$$

$$\frac{\Psi^2}{a^2h^3}\frac{1}{q'^2}\left\{\frac{\partial}{\partial(z',x')}[\psi',D'^2(\psi')] + \frac{2D'^2(\psi')G'(\psi')}{1-x'^2}\right\} + \frac{2U^2}{h}\frac{M'G'(M')}{q'^2(1-x'^2)}$$

$$+ \frac{2\Omega aU}{h}G'(M') = \gamma g(1-x'^2)\frac{\partial\rho'}{\partial x'} + \frac{K\Psi}{h^4}D'^4(\psi') \qquad (12)$$

$$\frac{\Psi}{a^2h}\frac{1}{q'^2}\frac{\partial(\psi',\rho')}{\partial(z',x')} = \frac{K_\rho}{h^2}\nabla'^2\rho', \qquad (13)$$

where

$$G' = x'\frac{\partial}{\partial z'} + \alpha\frac{1-x'^2}{q'}\frac{\partial}{\partial x'}$$

$$D'^2 = \frac{\partial^2}{\partial z'^2} + \alpha^2\frac{1-x'^2}{q'^2}\frac{\partial^2}{\partial x'^2}$$

$$\nabla'^2 = \frac{1}{q'^2}\left\{\frac{\partial}{\partial z'}\left(q'^2\frac{\partial}{\partial z'}\right) + \alpha^2\frac{\partial}{\partial x'}\left[(1-x^2)\frac{\partial}{\partial x}\right]\right\}.$$

The boundary conditions, in nondimensional form, are

$$\left.\begin{aligned}\psi' = \partial\psi'/\partial z' &= 0\\ M' &= 0\\ \partial\rho'/\partial z' &= 0\end{aligned}\right\}\text{ at } z' = 0,$$

and

$$\left.\begin{aligned}\psi' &= 0\\ q'^2\frac{\partial}{\partial z'}\left(q'^{-2}\frac{\partial\psi'}{\partial z'}\right) &= \frac{ah^2\tau_M}{\bar{\rho}K\Psi}q'(1-x'^2)^{\frac{1}{2}}\tau'_m\\ &= (\lambda/\epsilon)q'(1-x'^2)^{\frac{1}{2}}\tau'_m\end{aligned}\right\}\text{ at } z' = 1$$

$$\left.\begin{aligned} q'^2 \frac{\partial}{\partial z'}\left(q'^{-2}M'\right) &= \frac{h\tau_M}{\bar{\rho}KU} T' \\ &= \lambda T' \\ \frac{\partial \rho'}{\partial z'} = \frac{hQ_M}{\gamma\bar{\rho}K_\rho} Q' &= \lambda_1 Q' \end{aligned}\right\} \text{ at } z' = 1.$$

Here, λ, λ_1, and ϵ represent nondimensional constants.

The nonlinear term in (11) is of significance only if the zonal velocity component is a considerable fraction of the equatorial velocity Ωa of the earth's surface. As such high velocities are not present in the ocean, equation (11) must represent a balance essentially between the Coriolis force and the gradient of the zonal stress component. In order to express this balance by nondimensional terms of equal magnitude, the dimensional coefficients of (11) are chosen so that

$$2\Omega\Psi/h = KaU/h^2.$$

Similarly, it can be shown that (12) represents a balance mainly between the horizontal gradient of the force due to gravity and the vertical gradient of the Coriolis force, i.e., the zonal current is approximately geostrophic. The balance is expressed in nondimensional form by choosing

$$2\Omega aU/h = \gamma g.$$

The characteristic values of Ψ and U are, therefore,

$$\Psi = \gamma gK/(2\Omega)^2$$

$$U = \gamma ga/2\Omega.$$

The characteristic value of γ is chosen so that $\lambda_1 = 1$, i.e.,

$$\gamma = hQ_M/\bar{\rho}K_\rho.$$

Equations (10), (11), and (12) can be written in nondimensional form using the nondimensional parameters δ, ϵ, and μ where

$$\delta = \tfrac{1}{2}U/\Omega a$$

$$\epsilon = K/2\Omega h^2$$

$$\mu = \delta K/K_\rho .$$

The nondimensional equations, with primes omitted, are

$$\delta \frac{1}{q^2} \frac{\partial(\psi, M)}{\partial(z,x)} - G(\psi) = D^2(M) \tag{14}$$

$$\delta\epsilon^2 \frac{1}{q^2} \frac{\partial}{\partial(z,x)}[\psi, D^2(\psi)] + \frac{2D^2(\psi)G(\psi)}{q^2(1 - x^2)} + \left[1 + \frac{2\delta M}{q^2(1 - x^2)}\right]G(M)$$

$$= (1 - x^2)\frac{\partial\rho}{\partial x} + \epsilon^2 D^4(\psi) \tag{15}$$

$$\mu \frac{1}{q^2} \frac{\partial(\psi, \rho)}{\partial(z,x)} = \nabla^2\rho . \tag{16}$$

Using the values

$$a = 6371 \text{ km}$$

$$h = 4 \text{ km}$$

$$g = 10^3 \text{ cm sec}^{-1}$$

$$\Omega = 7.29\times10^{-5} \text{ sec}^{-1}$$

$$\bar{\rho} = 1.0 \text{ gm cm}^{-3}$$

$$\tau_M = 2 \text{ dyne cm}^{-2}$$

$$Q_M = 10^{-7} \text{ gm cm}^2 \text{ sec}^{-1}$$

$$K = 2\times10^4 \text{ cm}^2 \text{ sec}^{-1}$$

$$K_\rho = 25 \text{ cm}^2 \text{ sec}^{-1},$$

the characteristic dimensional and nondimensional constants are found to have the values

$$\Psi' = 1.50\times10^{6}\ m^{3}\ sec^{-1}$$
$$U = 6.89\ cm\ sec^{-1}$$
$$\gamma = 1.60\times10^{-3}$$
$$\alpha = 6.28\times10^{-4}$$
$$\delta = 7.42\times10^{-5}$$
$$\epsilon = 8.57\times10^{-4}$$
$$\mu = 5.93\times10^{-2}$$
$$\lambda = 5.81$$
$$K/K_{\rho} = 800.$$

The appropriate values of Q_M, K, and K_ρ are uncertain. The values chosen above are based on estimates given in Appendix II.

On the basis of the study of relative magnitudes of the terms in the equations, the following set of simplified equations is chosen for further analysis:

$$\delta \frac{1}{q^2} \frac{\partial(\psi,M)}{\partial(z,x)} - G(\psi) = D^2(M) \tag{17}$$

$$G(M) = (1 - x^2)\frac{\partial\rho}{\partial x} + \epsilon^2 D^4(\psi) \tag{18}$$

$$\mu \frac{1}{q^2} \frac{\partial(\psi,\rho)}{\partial(z,x)} = \nabla^2\rho. \tag{19}$$

This choice of equations involves the assumption that the inequalities

$$\delta\epsilon^2 \frac{1}{q^2} \frac{\partial}{\partial(z,x)}[\psi, D^2(\psi)] \ll 1$$

$$\frac{2\delta M}{q^2(1 - x^2)} \ll 1$$

hold in the ocean region under consideration.

6. Perturbation equations

As the system of equations chosen in the preceeding section is still too complex to treat analytically, the equations are reduced to a simpler form by the standard method of small perturbations. Considering α as an independent parameter, the equations corresponding to the limit $\alpha \to 0$ are determined. These equations, termed the zero-order equations, are independent of α and have the form

$$\epsilon^2 \frac{\partial^4 \psi_o}{\partial z^4} - x \frac{\partial M_o}{\partial z} = - (1 - x^2) \frac{\partial \rho_o}{\partial x} \tag{20}$$

$$\frac{\partial^2 M_o}{\partial z^2} + x \frac{\partial \psi_o}{\partial z} = \delta \frac{\partial(\psi_o, M_o)}{\partial(z,x)} \tag{21}$$

$$\frac{\partial^2 \rho_o}{\partial z^2} = \mu \frac{\partial(\psi_o, \rho_o)}{\partial(z,x)}, \tag{22}$$

with the boundary conditions

$$\left.\begin{aligned} \psi_o = \partial\psi_o/\partial z &= 0 \\ M_o &= 0 \\ \partial\rho_o/\partial z &= 0 \end{aligned}\right\} \text{ at } z = 0,$$

and

$$\left.\begin{aligned} \psi_o &= 0 \\ \partial^2\psi_o/\partial z^2 &= (\lambda/\epsilon)(1 - x^2)^{\frac{1}{2}}\tau_m \end{aligned}\right\} \text{ at } z = 1$$

$$\left.\begin{aligned} \partial M_0/\partial z &= \lambda T \\ \partial \rho_0/\partial z &= Q \end{aligned}\right\} \text{ at } z = 1.$$

For $\alpha \neq 0$, correction terms, i.e., the perturbation terms, which are functions of α must be added to the solutions of the zero-order equations. These correction terms may be developed into power series in α. As α is very small in the present model, the correction terms are small and will not change the qualitative features of the zero-order solutions. The correction terms for $\alpha \neq 0$ are not considered in the subsequent analysis.

In view of the amount of simplification required to arrive at a system of equations which can be studied analytically, it is of considerable interest to know which processes are represented by the zero-order equations (20), (21), and (22). It can be seen from equation (20) that the zonal flow is geostrophic wherever the term $\epsilon^2 \partial^4 \psi_0/\partial z^4$ is negligible. As this term is not negligible near the upper and lower boundaries of the ocean, the zonal flow here will not be geostrophic. The process of eddy diffusion is effective in the vertical direction only because the vertical gradients of density and angular momentum are much larger than the horizontal gradients. The zonal component of wind stress is transmitted almost unchanged to the ocean bottom because the transport of zonal angular momentum out of the ocean region by the meridional currents is negligible. The nonlinear term representing the transport of relative zonal angular momentum by the meridional currents is included in (21) so that its effect on the torque acting on the ocean bottom may be determined. The transport of mass by the meridional currents is significant because the parameter μ is not negligibly small.

In the subsequent equations the subscripts denoting the zero-order are omitted because only the zero-order equations are considered.

7. Currents driven by surface wind stress in a homogeneous ocean

The simplest flow described by the present model occurs when the ocean has a homogeneous density structure, i.e., when the mass flux across the ocean surface is zero.

The equations governing the motion in the homogeneous ocean are

$$\epsilon^2 \frac{\partial^4 \psi}{\partial z^4} - x \frac{\partial M}{\partial z} = 0 \tag{23}$$

$$\frac{\partial^2 M}{\partial z^2} + x \frac{\partial \psi}{\partial z} = \delta \frac{\partial(\psi, M)}{\partial(z, x)}. \tag{24}$$

These equations are of the boundary-layer type because the highest-order derivative is multiplied by a small parameter. A first approximation to the solution of (23) and (24) can be obtained by neglecting the nonlinear term. The linearized equations are

$$\epsilon^2 \frac{\partial^4 \psi}{\partial z^4} + x^2 \psi = \lambda x T \tag{25}$$

$$\frac{\partial M}{\partial z} + x\psi = \lambda T. \tag{26}$$

The boundary-layer solution of (25) and (26) in the upper boundary layer is

$$\psi = (\lambda T/x)\{1 - \exp[-k(1 - z)] \cos k(1 - z)\}$$

$$M = \lambda T/k\{1 + \tfrac{1}{2}\exp[- k(1 - z)][\cos k(1 - z) - \sin k(1 - z)]\}$$

and in the lower boundary layer is

$$\psi = (\lambda T/x)[1 - \exp(- kz)(\cos kz + \sin kz)]$$

$$M = (\lambda T/k)[1 - \exp(- kz) \cos kz],$$

where

$$k = (|x|/2\epsilon)^{\frac{1}{2}} \gg 1.$$

In the interior of the ocean, the solution is

$$\psi = \lambda T/x$$

$$M = \lambda T/k.$$

As long as the upper and lower boundary layers are separated, the boundary-layer solution may be expressed compactly as

$$\psi = (\lambda T/x)(1 - f) \tag{27}$$

$$M = (\lambda T/k)(1 - F), \tag{28}$$

where

$$f = \exp[- k(1 - z)]\cos k(1 - z) + \exp(- kz)(\cos kz + \sin kz) \tag{29}$$

$$\partial F/\partial z = - kf, \quad F(0,k) = 1.$$

The current system given by (27) and (28) is the same as the current system set up by a steady wind stress as described by Ekman (1905).

By substituting (27) and (28) into (24), the bottom torque T_B, including the contribution of the nonlinear term, may be computed. The bottom torque, to the first-order approximation in δ, is given by

$$T_B = T + \frac{\lambda\delta}{8k}\frac{d}{dx}\left(\frac{T}{x}\right)^2$$

and, depending on the distribution of T, can be greater or less than the surface torque. As $\delta\lambda/8k$ is very small in the present model, the bottom torque cannot differ appreciably from the surface torque, and no qualitative changes occur if the nonlinear terms are neglected altogether.

The current system set up by a meridional component of wind stress τ_m is confined to the upper layers of the ocean. For simplicity, τ_m is assumed to be zero in the subsequent analysis.

8. Currents driven by a mass flux across the ocean surface

If the surface wind stress is zero everywhere, the circulation set up by a mass flux across the ocean surface must satisfy the equations

$$\epsilon^2 \frac{\partial^4 \psi}{\partial z^4} + x^2 \psi = -(1 - x^2)\frac{\partial \rho}{\partial x} \tag{30}$$

$$\frac{\partial^2 \rho}{\partial z^2} = \mu\left(\frac{\partial \psi}{\partial z}\frac{\partial \rho}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \rho}{\partial z}\right) \tag{31}$$

$$\frac{\partial M}{\partial z} + x\psi = 0. \tag{32}$$

No exact, nontrivial solutions of these equations have been obtained because of the nonlinearity of (31). The formal boundary-layer equations are too complicated to solve. It is possible, however, to obtain approximate solutions which are valid in the interior of the ocean. In order to relate the interior density distribution to the surface mass flux, the boundary-layer equations are simplified by taking an average value of $\partial\psi/\partial z$ in the boundary layer and by neglecting smaller terms in the equations.

The boundary-layer solution of (30) is given with sufficient accuracy by

$$\psi = -\frac{1 - x^2}{x^2}\frac{\partial \rho}{\partial x}(1 - f),$$

where f is given by (29). This boundary-layer solution for ψ is obtained by neglecting the vertical variations of $\partial\rho/\partial x$ in the boundary layers. Using the value $3\pi/4k$ as a measure of the thickness of the upper boundary layer, the average value of $\partial\psi/\partial z$ in the upper boundary layer is estimated to be

$$\frac{\partial \psi}{\partial z} \simeq \frac{4k}{3\pi}\,\frac{1 - x^2}{x^2}\,\frac{\partial \rho}{\partial x}.$$

In deriving this estimate of $\partial\psi/\partial z$, the vertical variations of $\partial\rho/\partial x$ in the boundary layer are neglected.

The simplified boundary-layer equations for the density may be written

$$\frac{\partial^2\rho}{\partial z^2} \fallingdotseq \frac{4\mu k}{3\pi}\,\frac{1 - x^2}{x^2}\left(\frac{\partial\rho}{\partial x}\right)^2 . \tag{33}$$

The density gradient in the boundary layer is estimated by integrating (33) with respect to z again neglecting the vertical variations of $\partial\rho/\partial x$. The gradient is

$$\frac{\partial\rho}{\partial z} \fallingdotseq Q - \frac{4\mu k}{3\pi}\,\frac{1 - x^2}{x^2}\left(\frac{\partial\rho}{\partial z}\right)^2 (1 - z).$$

Below the boundary layer, i.e., at $z = 1 - 3\pi/4k$, the density gradient is

$$\frac{\partial\rho}{\partial z} \fallingdotseq Q - \mu\,\frac{1 - x^2}{x^2}\left(\frac{\partial\rho}{\partial x}\right)^2 , \tag{34}$$

and the density is

$$\rho \fallingdotseq \rho_{\text{surface}} + \frac{3\pi\mu}{8k}\,\frac{1 - x^2}{x^2}\left(\frac{\partial\rho}{\partial x}\right)^2 - \frac{3\pi}{4k}\,Q. \tag{35}$$

Obviously, these estimates of ρ and $\partial\rho/\partial z$ below the boundary layer are rough estimates only. In view of the uncertainty in choosing appropriate values of μ and k, the increased complexity of making more exact estimates of ρ and $\partial\rho/\partial z$ is probably not justified. A more detailed analysis of the boundary-layer equations shows that the estimates given above are qualitatively correct.

The relationships given by (34) and (35) are used to extend the interior density and density-gradient distributions continuously to the surface. Similar relationships are used in the lower boundary layer.

The equation governing the interior density distribution is derived by neglecting the term $\epsilon^2\partial^4\psi/\partial z^4$ in (30) and eliminating the stream function from (31). The equation is

$$\frac{\partial^2\rho}{\partial z^2} = \mu\left[\frac{\partial\rho}{\partial z}\frac{\partial}{\partial x}\left(\frac{1-x^2}{x^2}\frac{\partial\rho}{\partial x}\right) - \frac{1-x^2}{x^2}\frac{\partial\rho}{\partial x}\frac{\partial^2\rho}{\partial z\partial x}\right]. \tag{36}$$

As general solutions of (36) have not been found, the mass flux at the surface cannot be specified arbitrarily but must be chosen to be consistent with the interior density distribution given by special solutions of (36).

A new coordinate variable y is introduced so that

$$\frac{1-x^2}{x^2}\frac{\partial}{\partial x} = \frac{\partial}{\partial y},$$

that is,

$$y = \tfrac{1}{2}\log_e\left(\frac{1+x}{1-x}\right) - x.$$

Equation (36), in terms of y, becomes

$$g(y)\frac{\partial^2\rho}{\partial z^2} = \mu\left(\frac{\partial\rho}{\partial z}\frac{\partial^2\rho}{\partial y^2} - \frac{\partial\rho}{\partial y}\frac{\partial^2\rho}{\partial z\partial y}\right), \tag{37}$$

where

$$g(y) = \frac{1-[x(y)]^2}{[x(y)]^2}.$$

A solution of the form

$$\rho = \sigma(z)\chi(y)$$

is substituted into (37) and the resulting equation is written as

$$\frac{d^2\sigma}{dz^2}\Big/\sigma\frac{d\sigma}{dz} = \mu\left[\chi\frac{d^2\chi}{dy^2} - \left(\frac{d\chi}{dy}\right)^2\right]\Big/g\chi = -2\mu\nu, \tag{38}$$

where ν is a constant. There is no loss in generality in assuming

$\nu = 1$, so that the equation for σ becomes

$$\frac{d^2\sigma}{dz^2} = -2\mu\sigma\frac{d\sigma}{dz}. \tag{39}$$

A solution of (39) is

$$\sigma = c_o \operatorname{cotan} \mu c_o(z - z_o),$$

where z_o, c_o are arbitrary constants.

The equation for χ is

$$\left(\frac{d\chi}{dy}\right)^2 - \chi\frac{d^2\chi}{dy^2} = 2g\chi. \tag{40}$$

No exact, nontrivial solutions of (40) have been found. It is possible, however, to obtain an approximate solution, which is valid over a limited range of y, of the form

$$\chi = g_o\chi_o,$$

where

$$g_o = \exp(a_o + a_1 y + a_2 y^2).$$

The constants a_o, a_1, a_2 are chosen so that g/g_o is approximately equal to unity in the region under consideration. This may be done with considerable accuracy for an ocean lying poleward of 45^o because g(y) is asymptotic to an exponential function of y for large values of y. The equation for χ_o is

$$\left(\frac{d\chi_o}{dy}\right)^2 - \chi_o\frac{d^2\chi_o}{dy^2} = 2a_2\chi_o^2 + 2\frac{g}{g_o}\chi_o$$

$$\simeq 2a_2\chi_o^2 + 2\chi_o. \tag{41}$$

The first integral of (41) is

$$\left(\frac{d\chi_o}{dy}\right)^2 = 4\chi_o - 4a_2\chi_o^2 \log_e \chi_o + 4c_1\chi_o^2, \tag{42}$$

where c_1 is an arbitrary constant. Thus, the solution of (41), in inverse form, may be written

$$y - y_o = \frac{1}{2}\int_0^{\chi_o} (\xi - a_2\xi^2 \log_e \xi + c_1\xi^2)^{-\frac{1}{2}} d\xi, \tag{43}$$

where y_o is an arbitrary constant. The integral must be evaluated numerically.

An approximate solution of (37) is, therefore, given by

$$\rho(z,y) = - c_o g_o \chi_o \cot\text{an}[\mu c_o(z_o - z)]. \tag{44}$$

Only z_o and c_o of the undetermined constants are specified by the boundary conditions. The constants a_o, a_1, a_2 depend on the region chosen for the ocean, and the constants y_o, c_1 determine a particular horizontal distribution of surface mass flux. The details involved in relating the boundary conditions to the interior density distribution are given in Appendix III.

9. Currents driven by both surface wind stress and surface mass flux

If the circulation is influenced by wind stress and surface mass flux, the equations governing the circulation and the density distribution have the form

$$\epsilon^2 \frac{\partial^4\psi}{\partial z^4} + x^2\psi = \lambda xT - (1 - x^2)\frac{\partial\rho}{\partial x} \tag{45}$$

$$\frac{\partial^2\rho}{\partial z^2} = \mu\left(\frac{\partial\psi}{\partial z}\frac{\partial\rho}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\rho}{\partial z}\right) \tag{46}$$

$$\frac{\partial M}{\partial z} + x\psi = \lambda T. \tag{47}$$

The boundary-layer solution of (45) may be written

$$\psi = \left(\frac{\lambda T}{x} - \frac{1 - x^2}{x^2}\frac{\partial\rho}{\partial x}\right)(1 - f),$$

where f is given by (29). The average value of $\partial\psi/\partial z$ in the upper boundary layer is approximately

$$\frac{4k}{3\pi}\left(\frac{\lambda T}{x} - \frac{1 - x^2}{x^2}\frac{\partial\rho}{\partial x}\right),$$

so that the simplified boundary-layer equation for the upper boundary layer is

$$\frac{\partial^2\rho}{\partial z^2} \doteq -\frac{4\mu k}{3\pi}\left(\frac{\lambda T}{x} - \frac{1 - x^2}{x^2}\frac{\partial\rho}{\partial x}\right)\frac{\partial\rho}{\partial x}. \qquad (48)$$

An approximate value of $\partial\rho/\partial z$ below the upper boundary layer is found by integrating (48) with respect to z neglecting the variations of $\partial\rho/\partial x$ in the boundary layer. The value is

$$\frac{\partial\rho}{\partial z} \doteq Q + \mu\left(\frac{\lambda T}{x} - \frac{1 - x^2}{x^2}\frac{\partial\rho}{\partial x}\right)\frac{\partial\rho}{\partial x}. \qquad (49)$$

Elimination of the stream function from (46) yields the following equation for the interior density distribution:

$$\frac{\partial^2\rho}{\partial z^2} + \mu\lambda\left(\frac{d}{dx}\frac{T}{x}\right)\frac{\partial\rho}{\partial z} = \mu\left[\frac{\partial\rho}{\partial z}\frac{\partial}{\partial x}\left(\frac{1 - x^2}{x^2}\frac{\partial\rho}{\partial x}\right) - \frac{1 - x^2}{x^2}\frac{\partial\rho}{\partial x}\frac{\partial^2\rho}{\partial z\partial x}\right]. \qquad (50)$$

As the term $\lambda d(T/x)/dx$ is an order of magnitude larger than the horizontal density gradient in the interior of the ocean, the terms involving $\partial\rho/\partial x$ in (50) are neglected. The resulting simplified equation is linear and a solution may be chosen of the form

$$\rho(z,x) = \rho_1(x)\left\{\exp\left[\mu\lambda\left(\frac{d}{dx}\frac{T}{x}\right)(z_o - z)\right] - d_o\right\}, \qquad (51)$$

where z_o, d_o are arbitrary constants.

The interior density distribution, given by (51), is fitted to the density distribution in the upper boundary layer by requiring that the equation

$$-\mu\lambda\left(\frac{d}{dx}\frac{T}{x}\right)\rho_1 = Q + \mu(1 - d_o)\left[\frac{\lambda T}{x} - (1 - d_o)\frac{1 - x^2}{x^2}\frac{\partial\rho_1}{\partial x}\right]\frac{\partial\rho_1}{\partial x}, \qquad (52)$$

obtained by substituting (51) into (49) and setting $z = z_o$, be satisfied at $z = 1 - 3\pi/4k$.

Equation (52), together with an analogous equation for the lower boundary layer, is solved numerically for ρ_1.

The interior density distribution, as given by (51), is stable if $d(T/x)/dx$ is negative, i.e., if the vertical component of velocity induced by the wind torque is positive. For simplicity, the wind torque is assumed to be of the form

$$T = - x(b_o + b_1 x)$$

so that

$$d(T/x)/dx = - b_1,$$

where b_1 is a positive constant.

10. Numerical examples

The particular distributions of wind stress and mass flux chosen in the previous sections are not intended to resemble the actual distribution of these quantities over the ocean surface. The wind stress and mass flux distributions are chosen to simplify the analytical treatment and the numerical computations and to give a density field resembling qualitatively the density field associated with the Antarctic Circumpolar Current when the combined influence of wind stress and mass flux is considered.

The ocean region is chosen to lie between 70.1°S and 45°S, roughly the region occupied by the Antarctic Circumpolar Current. In this region, an example of each of the following three types of flow is considered in detail: (A) the circulation in a homogeneous ocean set up by a distribution of wind stress over the ocean surface, (B) the circulation set up by a distribution of surface mass flux alone, (C) the circulation set up by the combination of surface wind stress and surface mass flux.

Streamlines of the meridional circulation illustrating the three different types of flow are shown in meridional cross-sections in figures (1), (2), and (3). The streamlines are labeled in units of $\lambda\psi (= 8.72 \times 10^6 \ m^3 \ sec^{-1})$.

The density structure in figures (2) and (3) is indicated by isopycnals (broken lines) which are labeled in units of $\rho\gamma (= 1.6 \times 10^{-3} \ gm \ cm^{-3})$. For purposes of comparison, an example of the observed density structure associated with the Antarctic Circumpolar Current is shown in figure (4). Here, the broken lines represent lines of constant σ_t.[1]

The vertical distributions of the zonal velocity component for the three types of circulation are compared in figure (5). The comparison is made at 45°S where the zonal velocity component for each type of circulation reaches a maximum. The horizontal distributions of surface wind stress and surface mass flux in the computations are shown in figure (6).

The total volume transport through the meridional cross-section for the three types of flow is found to be as follows:

(A) $112 \times 10^6 \ m^3 \ sec^{-1}$, for wind stress alone
(B) $-755 \times 10^6 \ m^3 \ sec^{-1}$, for mass flux alone
(C) $433 \times 10^6 \ m^3 \ sec^{-1}$, for wind stress and mass flux.

11. Discussion

The present model consists of an ocean zone that is uniform and uninterrupted in the longitudinal direction, hence, a toroid. The model ocean is of constant depth, is bounded below by a level bottom and above by a free surface, and is located poleward of 45°S. The model is open toward the equator and the

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[1] The quantity σ_t is equal to $10^3(\rho - 1)$, where ρ represents the density of sea water brought isothermally to atmospheric pressure.

pole. It is assumed that no flow occurs across the poleward side.

The circulation is driven by wind stress and by the flux of heat and water across the ocean surface. Exchanges of heat and water across the surface give rise to changes in the density through thermal expansion and the concentration or dilution of the salts dissolved in sea water. The presence of density differences in the ocean influences strongly the types of circulation which can exist. The thermodynamic processes which influence the circulation are taken into account in a simplified manner. The volume changes associated with the density changes are neglected. The actual fluxes of heat and water are replaced in the model with an equivalent net flux of mass. In the interior of the ocean, mass is distributed by convection and diffusion.

Simple distributions of surface wind stress and surface mass flux are chosen and the steady three-dimensional circulation which can be induced is illustrated by the following three examples: (A) the circulation set up by a west wind alone, (B) the circulation set up by surface mass flux alone, (C) the circulation set up by the combination of wind stress and surface mass flux.

The meridional circulation of type A, resulting from an eastward stress on the surface, is illustrated in figure 1. The ocean is homogeneous in density, and the steady-state meridional currents are confined to relatively shallow layers near the upper and lower boundaries of the ocean. These meridional currents transport a negligible amount of angular momentum and no net mass into the model, so that a relatively simple meridional circulation can satisfy the conditions necessary for a steady state to exist. The condition of zero flow across the poleward limit of the ocean region is satisfied by choosing the wind stress to be zero at this limit. The zonal current is uniform with depth except in the boundary layers. For the wind-stress distribution shown in figure 6 the eastward volume transport of the zonal current is $112 \times 10^{6}\ m^{3}\ sec^{-1}$.

If there is a positive upward flux of mass through the ocean surface, nonhomogeneous density distributions can exist in the ocean, as illustrated in figures 2 and 3. The mass lost through the ocean surface must be replaced by meridional currents. In the circulations of type B and C, the mass lost through the surface is replaced by a relatively strong, shallow, poleward current flowing along the bottom of the ocean. In type B, the density decreases along streamlines only in the poleward current, except in the surface layer, while in type C, the density decreases along streamlines throughout the entire region. In the absence of wind stress, the mass flux induces a strong westward zonal transport totaling 755×10^6 m^3 sec^{-1}. The presence of eastward wind stress increases the poleward current and, consequently, the mass transport in the bottom layer to such an extent that the horizontal gradient of density throughout most of the ocean is opposite in sign to the gradient induced by mass flux alone. The total eastward transport is thus larger than the transport due to the wind stress alone, amounting to a total of 433×10^6 m^3 sec^{-1}.

The present model makes possible a more quantitative estimate of the influence of thermodynamic processes on oceanic circulation. The examples of the circulation discussed here are of a very special nature and have been obtained under restrictive assumptions so that quantitative application of results based on these examples to the actual ocean is not justified. The examples are given to illustrate some general features of the present model.

A characteristic feature of the present model is that whenever the eddy flux of mass or of angular momentum is nondivergent, the meridional component of velocity is negligibly small. Thus, if the present model can be validly applied to the actual ocean, the conclusion is reached that any steady meridional current must be accompanied by strong diffusion or mixing processes which change the density and the angular momentum of the water along a particular streamline. The strongest meridional currents develop, according to the present model, near the boundaries of the ocean where the eddy stress, i.e., the momentum flux, can attain a relatively high value.

The total zonal transport of water is greatly influenced by the presence of surface mass flux, as can be seen by comparing the total zonal transports for examples A and C. Thus, it would be incorrect to explain the zonal transport of the Antarctic Circumpolar Current, which is computed from the observed density distribution, in terms of the surface wind stress alone. According to the present model, a considerable range of values of the zonal transport is obtained for the same wind stress by varying the distribution of surface mass flux.

It has not been found possible to justify the assumption that processes occurring outside of the ocean region under consideration have little or no influence on the circulation in the region considered.

The solutions obtained are not valid outside of the region. Furthermore, no stable density distributions have been found for the regions where there is a positive downward flux of mass through the ocean surface.

The meridional circulation shown in the diagram differs considerably from the circulation believed to exist in the Antarctic Circumpolar Current. The difference cannot be attributed to oversimplified boundary conditions used in computing the circulation, but must be attributed to the failure of the model to simulate the processes occurring in the actual ocean.

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Appendix I. The mass-transfer equation

The density of sea water is a function of essentially three variables; temperature θ, salinity S, and pressure p. As the dependence of density on pressure is not dynamically significant, it is sufficient for the purposes of the present study to consider the density as a function of temperature and salinity only.

The heat content H and the salinity S are conservative quantities in the ocean, and it is assumed that the equations governing their distribution are of the form

$$DH/Dt = K_\theta \nabla^2 H$$

$$DS/Dt = K_S \nabla^2 S,$$

where D/Dt represents the total time derivative, K_θ and K_S the constant, isotropic kinematic eddy coefficients of heat conductivity and salt diffusivity respectively. The compressibility of the water is neglected in choosing the equations to be of this form.

The heat content H, relative to an arbitrary reference point, can be expressed in terms of temperature by the formula

$$H = c_p(\theta - \theta_o),$$

where c_p, the specific heat, is assumed to be constant.

If the density is a linear function of the temperature and salinity of the form

$$\rho = \bar{\rho}[1 - \alpha_\theta(\theta - \theta_o) + \alpha_S(S - S_o)],$$

where α_θ and α_S are constants, and if the eddy coefficients K_θ and K_S are equal, the two conservation equations for heat and salt can be combined into a single equation of the form

$$D\rho/Dt = K_\rho \nabla^2 \rho,$$

which represents the conservation of mass under convective and diffusive processes.

The actual relationship between density, temperature, and salinity may be approximated by

$$\rho = \rho_o[1 - \alpha_\theta(\theta - \theta_o) + \alpha_S(S - S_o)],$$

where α_θ and α_S are functions of both temperature and salinity, and where ρ_o, θ_o, and S_o are constant reference values. The coefficients α_θ, α_S can be calculated from tables. Using the auxiliary condition

$$\frac{\alpha_\theta(\theta,S) - \alpha_\theta(\theta,S_o)}{S - S_o} = - \frac{\alpha_S(\theta,S) - \alpha_S(\theta_o,S)}{\theta - \theta_o}$$

and the reference values $\theta_o = 10$ C, $S_o = 35‰$, and $\rho_o = 1.027$, the coefficient α_θ was found to vary between 0.96×10^{-4} C^{-1} and 2.06×10^{-4} C^{-1} for temperature between -2°C and 18°C and salinities between 32‰ and 37‰. Over the same range of temperature and salinity the coefficient α_S was found to vary between 0.750 and 0.776.

The values of α_θ and α_S used in the present report are

$$\alpha_\theta = 1.00 \times 10^{-4} \text{ C}^{-1}$$

$$\alpha_S = 0.76.$$

The flux of mass across the ocean surface may be expressed in terms of the flux of heat and the flux of water across the surface. The flux of water is equal to the difference between the rates of evaporation E and the precipitation P. If E - P is negative, i.e., water is added to the ocean, the salt content is decreased in the surface layers. This decrease in salinity is interpreted as being due to a virtual flux of salt Q_S across the ocean surface equal to $\overline{\rho S}(E - P)$, where $\overline{S}$ is the mean surface salinity.

The boundary condition for the density at the ocean surface may, therefore, be written

$$K_\rho \frac{\partial \rho}{\partial r} = \frac{1}{\bar{\rho}} Q = \frac{1}{\bar{\rho}}\left(- \alpha_\theta K_\theta \frac{\partial \theta}{\partial r} + \alpha_S K_S \frac{\partial S}{\partial r}\right) = \frac{1}{\bar{\rho}}\left(- \frac{\alpha_\theta}{c_p} Q_\theta + \alpha_S Q_S\right),$$

where Q, Q_θ, and Q_S are the fluxes of mass, heat, and salt respectively.

It is difficult to estimate with any degree of accuracy the values of Q_θ and Q_S. Generally speaking, the ocean loses heat to the atmosphere in the region of the Antarctic Circumpolar Current but, because the precipitation exceeds evaporation, the density at the ocean surface remains lower than the density at greater depths. Assuming that the maximum net precipitation rate is of the order of 130 cm year^{-1} and the minimum net heat loss is of the order of 10 cal cm^{-2} day^{-1}, and using the average surface salinity of 34‰, the values of Q_θ and Q_S are found to be

$$Q_\theta = - 1.2 \times 10^{-4} \text{ cal cm}^{-2} \text{ sec}^{-1}$$

$$Q_S = - 1.4 \times 10^{-7} \text{ gm cm}^{-2} \text{ sec}^{-1}.$$

Thus, an estimate of the maximum mass flux Q_M, according to the definition given above, is

$$Q_M = - (\alpha_\theta/c_p)Q_\theta + \alpha_S Q_S \fallingdotseq 10^{-7} \text{ gm cm}^{-2} \text{ sec}^{-1}.$$

This value of Q_M is assumed to be representative of the actual maximum mass flux across the ocean surface.

Appendix II. The kinematic eddy coefficients

The boundary conditions in nondimensional form for the zonal angular momentum and the density at the ocean surface are

$$q^2 \frac{\partial}{\partial z}(q^{-2}M) = \frac{h\tau_M}{\rho K U} T = \lambda T$$

$$\frac{\partial \rho}{\partial z} = \frac{hQ_M}{\rho K_\rho \gamma} Q = Q,$$

where τ_M and Q_M are maximum values of the wind stress and mass flux respectively.

An estimate of Q_M, given in appendix I, is 10^{-7} gm cm^{-2} sec^{-1}. The value of 2 dyne cm^{-2} is chosen for τ_M as being appropriate for the ocean region containing the Antarctic Circumpolar Current.

In order to have density differences in the present model comparable with the observed density differences in the region of the Antarctic Circumpolar Current, the characteristic fractional range of density variation γ must have a value comparable with the value obtained in the actual ocean. In the present model, γ is given by $hQ_M/\rho K_\rho$. Thus, for $K_\rho = 25$ cm^2 sec^{-1}, the value γ is found to be 1.60×10^{-3}. This value of γ is of the same magnitude as the fractional range of density variation observed in the ocean. A higher value of K_ρ, e.g., $K_\rho =$ 100 cm^2 sec^{-1}, would give almost negligible density difference according to the present model.

The eddy viscosity K is chosen to have the value 2×10^4 cm^2 sec^{-1}. A smaller value of K would give excessive velocities throughout the ocean. A larger value of K would give smaller velocities but, would also give the result that the circulation is determined primarily by the mass flux across the ocean surface. This does not appear to be the case in the actual ocean.

The necessity of assigning a large value to the ratio K/K_ρ in order to obtain reasonable results from the present model reflects the lack of a more suitable mechanism to counteract the wind torque applied at the ocean surface.

Appendix III. Boundary conditions (B)

The approximate equations relating the interior density gradients to the density gradients in the boundary layers are

$$\frac{\partial \rho}{\partial z} \simeq Q - \frac{\mu}{g}\left(\frac{\partial \rho}{\partial y}\right)^2 \qquad \text{at } z = 1 - 3\pi/4k,$$

$$\frac{\partial \rho}{\partial z} \simeq - \frac{\mu}{g}\left(\frac{\partial \rho}{\partial y}\right)^2 \qquad \text{at } z = \pi/k.$$

These equations determine the magnitude of the arbitrary constants in the solution for the interior density distribution.

The derivatives $\partial\rho/\partial z$, $\partial\rho/\partial y$ may be written

$$\frac{\partial \rho}{\partial z} = c_o[\cot\mu c_o(z_o - z) + \tan\mu c_o(z_o - z)]\rho$$

$$\frac{\partial \rho}{\partial y} = \frac{1}{g_o\chi_o}\frac{dg_o\chi_o}{dy}\rho, \quad g_o = \exp(a_o + a_1 y + a_2 y^2),$$

so that at $z = \pi/k$,

$$c_o[\cot\mu c_o(z_o - \pi/k) + \tan\mu c_o(z_o - \pi/k)]\rho = - \frac{\mu}{gg_o^2\chi_o^2}\left(\frac{dg_o\chi_o}{dy}\right)^2\rho^2.$$

Substitution for ρ yields the equation

$$\tan^2\mu c_o(z_o - \pi/k) = \frac{1}{gg_o\chi_o}\left(\frac{dg_o\chi_o}{dy}\right)^2 - 1.$$

This equation is satisfied approximately by choosing the arbitrary constant c_1 of (42) so that the variation of $[d(g_o\chi_o)/dy]^2/gg_o\chi_o$ is minimized. A relationship between z_o and c_o is then obtained using the average value of $[d(g\chi_o)/dy]^2/gg_o\chi_o$ in the region under consideration.

The mass flux Q across the ocean surface is determined from the equation connecting the interior density gradients to the density gradients in the upper boundary layer, i.e.,

$$Q = \frac{\partial \rho}{\partial z} + \frac{\mu}{g}\left(\frac{\partial \rho}{\partial y}\right)^2, \quad z = 1 - \frac{3\pi}{4k}.$$

Substitution for ρ yields

$$Q = -\mu g_o \chi_o c_o^2 \left\{ 1 - \left[\frac{\cot \mu c_o(z_o - 1 + 3\pi/4k)}{\cot \mu c_o(z_o - \pi/k)} \right]^2 \right\}.$$

The requirement that Q has a unit negative extremum value yields a second relation between z_o and c_o so that both constants can be determined.

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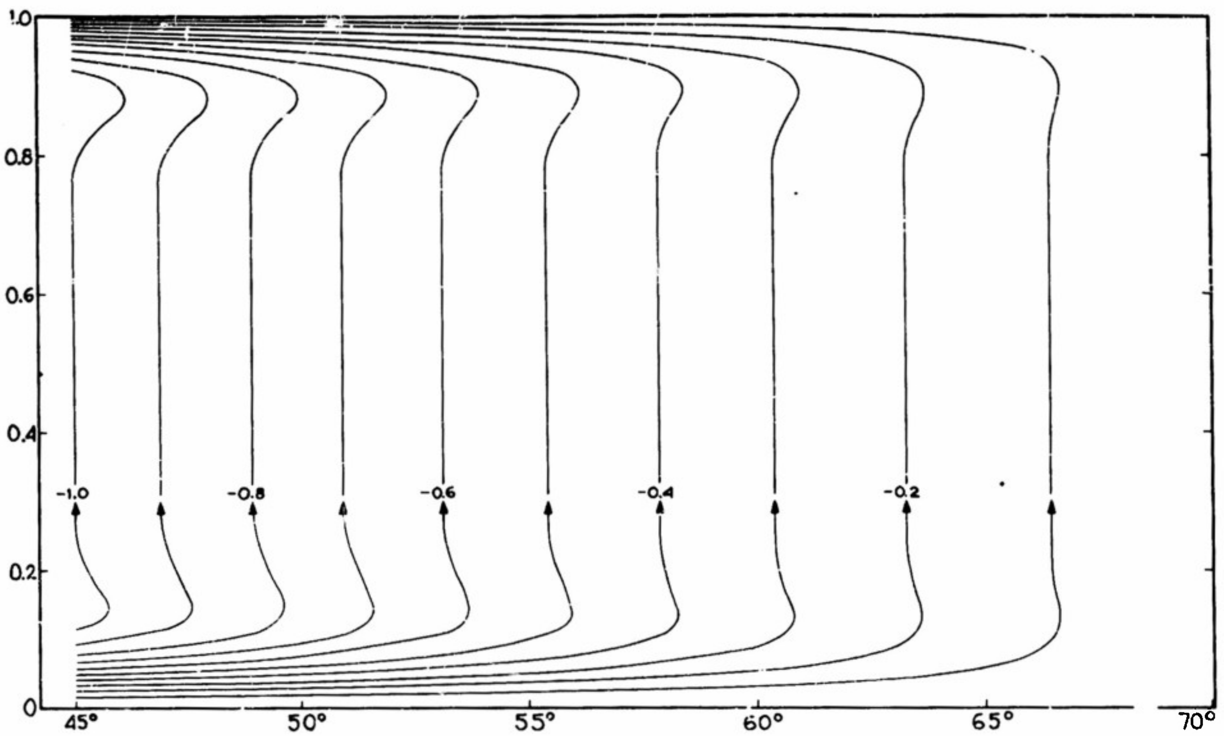

Fig. 1. (A) Meridional section showing streamlines of the meridional component of the wind-induced circulation in a homogeneous ocean. The streamlines are labeled in units of $\lambda\psi$ (= 8.72×10^{6} m^{3} sec^{-1}).

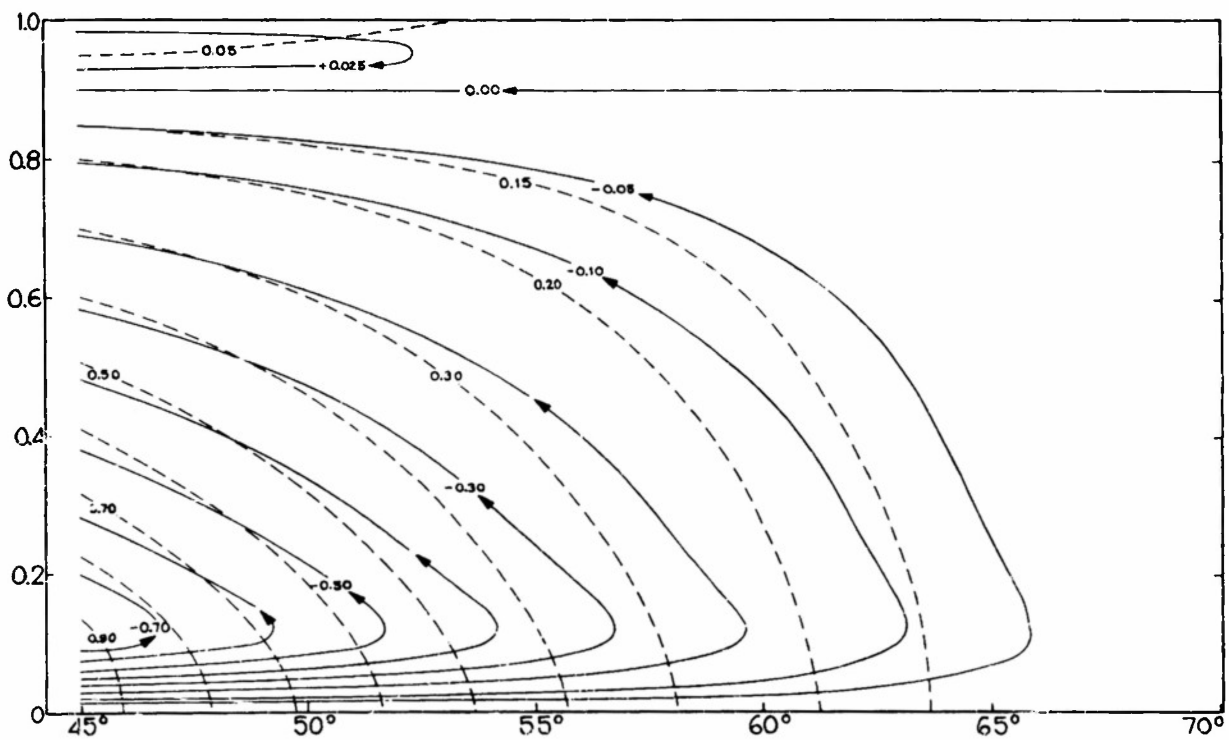

Fig. 2. (B) Meridional section showing streamlines of the meridional component of the circulation induced by a mass flux across the ocean surface. The streamlines are labeled in units of $\lambda\psi$. The broken lines indicate isopycnals and are labeled in units of $\beta\gamma$ (= 1.6×10^{-3} gm cm^{-3}).

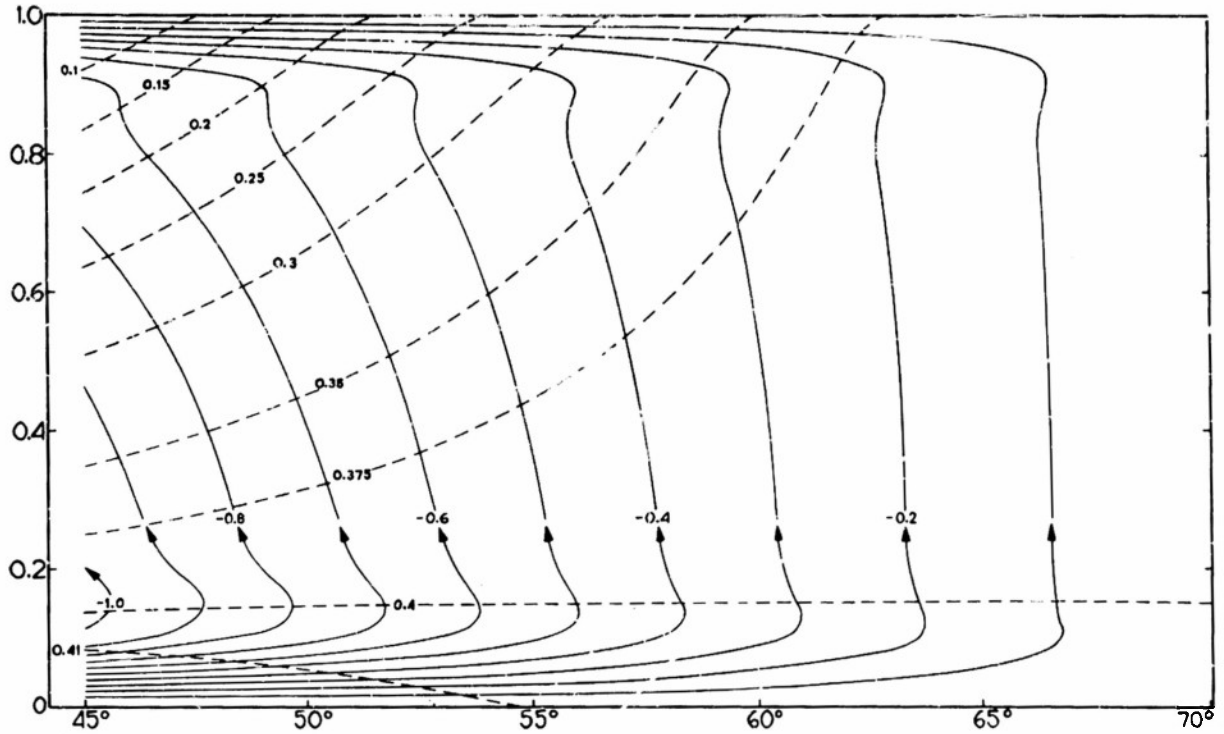

Fig. 3. (C) Meridional section showing streamlines of the meridional component of the circulation induced by the combination of surface wind stress and surface mass flux. The streamlines are labeled in units of $\lambda\psi$. The isopycnals are labeled in units of $\delta\gamma$ and are indicated by broken lines.

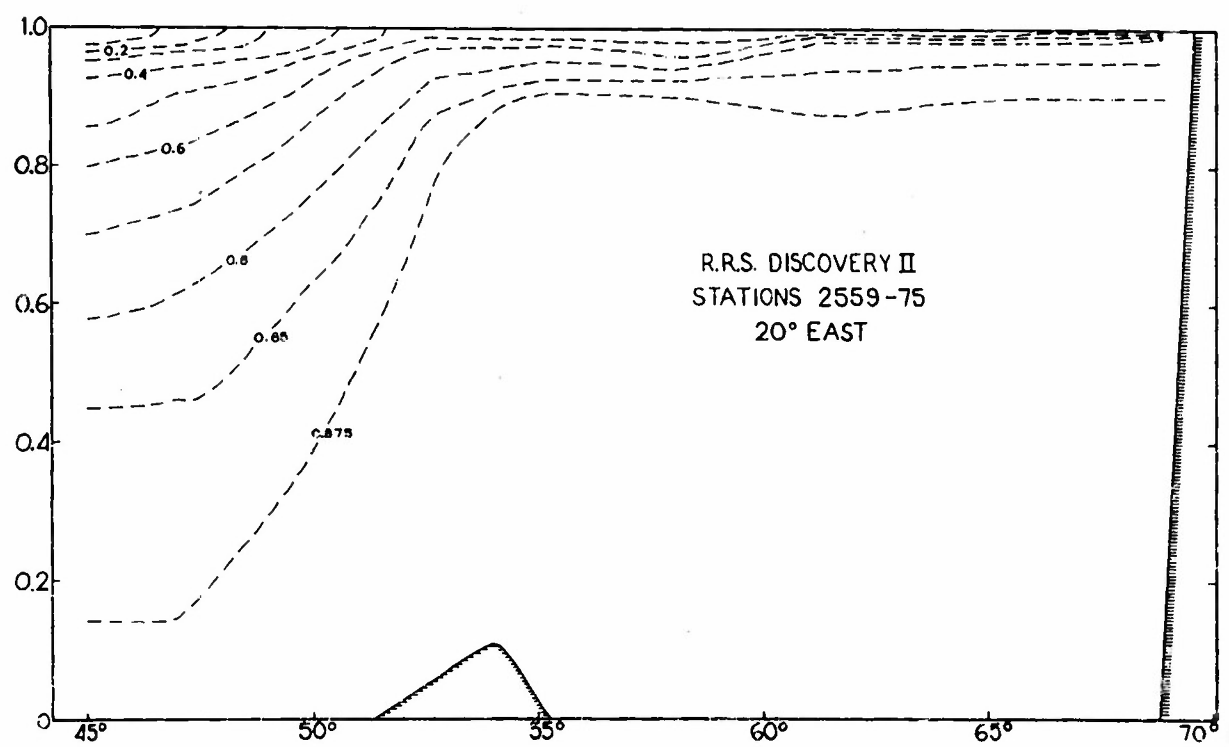

Fig. 4. A meridional cross-section showing the distribution of σ_t as inferred from observations taken along the meridian at approximately 20° E long. The σ_t lines are labeled in units of $\delta\gamma$.

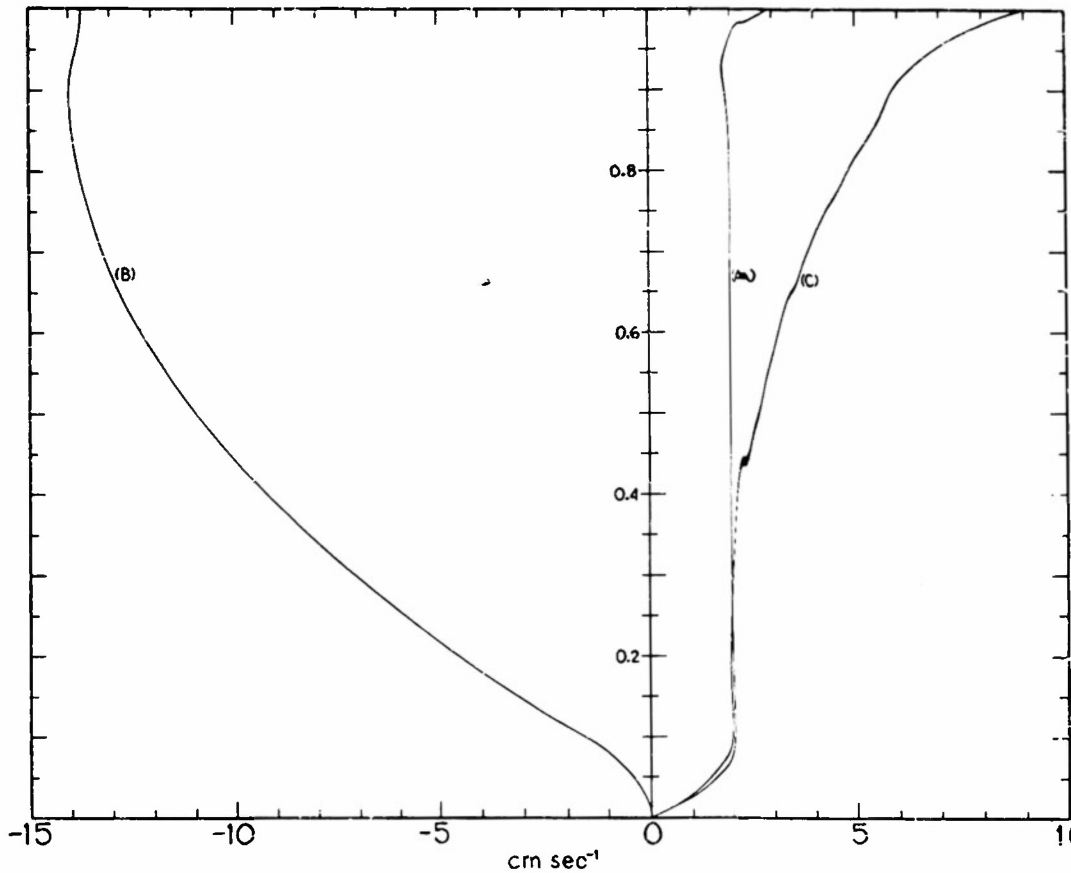

Fig. 5. The distributions of the zonal velocity component (positive eastwards) for the circulations (A), (B), and (C) along a vertical line at 45° S. The height unit is 4 km.

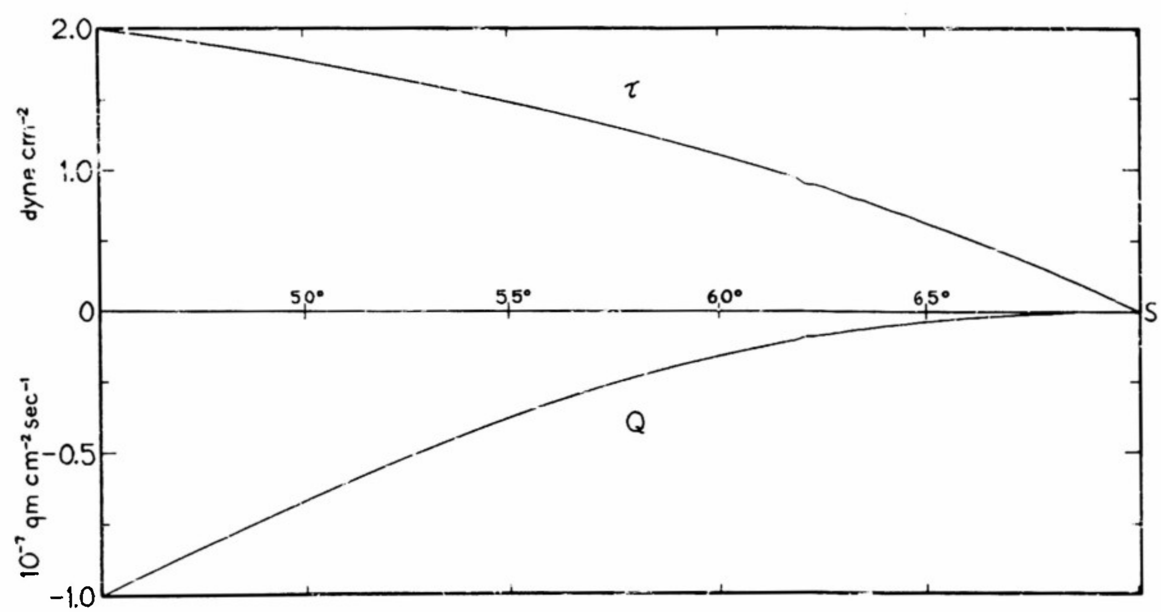

Fig. 6. The horizontal distributions of the surface wind stress τ (positive eastwards) and the surface mass flux Q used in computing the different types of circulation. A negative surface mass flux implies removal of mass from the ocean.

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